CURRENTS AND FIELDS IN AN ASYMMETRICALLY DRIVEN COAXIAL LINE

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Abstract: Cylindrical plasma pinch experiments are conveniently done in coaxial geometry. However, when a coaxial line is driven by a nonaxisymmetric source of current, there result asymmetries in the magnetic field in the coaxial line which in turn will drive Rayleigh-Taylor instabilities. In this paper we calculate the field asymmetry.

Introduction

Our basic goal is to vaporise and then implode a thin cylindrical metal foil by means of a very large, rapidly rising, current pulse. The resulting shell of plasma is compressed as a z-pinch. Because of the geometry of the foil it is natural to feed it by means of a coaxial line. The foil then simply forms a section of the inner conductor of this line.

In ideal coaxial geometry, the z-current (current parallel to the axis of the coaxial line) and the resulting B_θ-field are both perfectly axially symmetric. At present, however, our current sources are parallel plate generators. Connecting such a generator to a coaxial line leads of necessity to a departure from axial symmetry of the whole system. The result is that the current density and the magnetic field at the foil are no longer axisymmetric and this will drive a Rayleigh-Taylor instability (a flute mode) of the plasma shell. Such an instability could, if large enough, severely limit the maximum compression and temperature achieved in the pinch. Before the Rayleigh-Taylor problem can be undertaken, we must know the details of the driving pressure asymmetry, and this is the subject of this paper.

The Magnetic Field in the Dielectric

In our problem it will be convenient both in the parallel plate sections and in the coaxial sections, to work in cylinder coordinates (r, θ, z). Then the plate conductors will have the equation $z = a$ constant whereas on cylindrical conductors $r = a$ constant. We shall represent the magnetic field in the dielectric region between conductors as

$$\vec{B} = 2I/r\ \vec{e}_\theta + \nabla\phi \quad , \tag{1.1}$$

where $\vec{e}_\theta$ is a unit vector in the θ-direction and I is the total z-current flowing inside the radius r, and ϕ is a potential function describing the asymmetry. We have neglected the displacement current, i.e., the capacitative impedance. We shall not, however, assume that $\partial B/\partial t = 0$. As the divergence of B must vanish, we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad . \tag{1.2}$$

This, plus appropriate boundary conditions, will determine ϕ and thence B.

First let us consider the region between two parallel plates, the upper located at $z = z_0$ and the lower at $z = z_0 - h$. The dielectric thickness h will later be assumed to be small. The boundary condition on the magnetic field, namely that $B_z = 0$ at $z = z_0$ and at $z = z_0 - h$, is trivially satisfied by the solution

$$\phi = \sum_{m=1}^{\infty} [A_m(r/r_0)^m - B_m(r_0/r)^m] \sin m\theta \quad , \tag{1.3}$$

where A_m and B_m are arbitrary functions of time alone and r_0 is a typical (constant) radius, introduced for convenience. Using Eq. (1.3), the magnetic field components are

$$B_r = \sum_{m=1}^{\infty} \frac{m}{r} [A_m(r/r_0)^m + B_m(r_0/r)^m] \sin m\theta \quad ,$$

$$B_\theta = 2I/r + \sum_{m=1}^{\infty} \frac{m}{r} [A_m(r/r_0)^m - B_m(r_0/r)^m] \cos m\theta \quad , \tag{1.4}$$

$$B_z = 0$$

between the parallel plates.

Between a pair of coaxial cylinders $r = a$ and $r = b$ $(b = a+h)$, and assuming the same angular symmetry as in Eq. (1.3), we may write the solution as

$$\phi = \sum_{m=1}^{\infty} [C_m^* e^{\lambda z} - D_m^* e^{-\lambda z}] Z_m(\lambda r) \sin m\theta \quad , \tag{1.5}$$

where $Z_m(\lambda r)$ is an arbitrary linear combination of the Bessel functions J_m and Y_m and C_m^*, D_m^* are arbitrary functions of time. Strictly speaking Eq. (1.5) should also include a summation over λ; we shall take care of that presently. The boundary condition that B_r vanish at $r = a$, the inner cylinder wall, is taken care of by choosing

$$Z_m(\lambda r) = Y_m'(\lambda a)\, J_m(\lambda r) - J_m'(\lambda a)\, Y_m(\lambda r) \quad ; \tag{1.6}$$

then $Z_m'(\lambda a) = 0$. Here primes mean derivatives with respect to the argument, λr.

Next we must satisfy the boundary condition that B_r vanish at $r = b = a+h$. It is here that we make use of the fact that h is small by Taylor expanding,

$$Z_m'(\lambda b) = Z_m'(\lambda a) + \lambda h\, Z_m''(\lambda a) + \tfrac{1}{2}(\lambda h)^2\, Z_m'''(\lambda a) + \cdots \quad . \tag{1.7}$$

Now we can evaluate the derivates on the right hand side by noting that $Z_m(\rho)$ satisfies Bessel's equation and that $Z_m'(\lambda a)$ vanishes by its construction, In this fashion we obtain the expansion

$$Z_m'(\lambda b) = Z_m(\lambda a) \left\{ \left(\frac{m^2}{\lambda^2 a^2} - 1\right)(\lambda h) + \frac{1}{2\lambda a}\left(\frac{3m^2}{\lambda^2 a^2} - 1\right)(\lambda h)^2 + \cdots \right\} \quad . \tag{1.8}$$

When λh is sufficiently small, we can break off the sum in the curly brackets after a modest number of terms and satisfy the boundary conditions $Z_m'(\lambda b) = 0$

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by equating this bracketted expression to zero. To the order actually written out, this determines λ as

$$\lambda = (m/a)\ [1 - \tfrac{1}{2}(h/a) + O(h^2/a^2)] \quad . \tag{1.9}$$

What is required for this to work is that (h/a) must be small. If we define the mean radius ρ as

$$\rho \equiv a(1 + \frac{h}{2a}) = \tfrac{1}{2}(a + b) \quad , \tag{1.10}$$

then Eq. (1.9) can be written

$$\lambda = m/\rho \tag{1.11}$$

and Eq. (1.8) states that $Z_m'(mb/\rho) = O(m[h/\rho]^3)$, so our boundary condition is satisfied and for all $b = a + h > r > a$, $Z_m(mr/\rho) = Z_m(m)$, both to high precision. We can, therefore, replace the radial functions by constants and define new arbitrary functions which depend only on time

$$C_m(t) = C_m^* Z_m(m) \quad , \quad D_m(t) = D_m^* Z_m(m) \quad . \tag{1.12}$$

The fields are then given by

$$\begin{aligned} B_r &= 0 \quad , \\ B_\theta &= 2I/r + \sum_{m=1}^{\infty} (m/r)\ [C_m e^{mz/\rho} - D_m e^{-mz/\rho}] \cos m\theta \quad , \\ B_z &= \sum_{m=1}^{\infty} (m/\rho)\ [C_m e^{mz/\rho} + D_m e^{-mz/\rho}] \sin m\theta \quad . \end{aligned} \tag{1.13}$$

to high precision.

Boundary Conditions at the Junction of Two Lines

In Fig. 1 we have sketched the junction of a parallel plate line with a coaxial line.

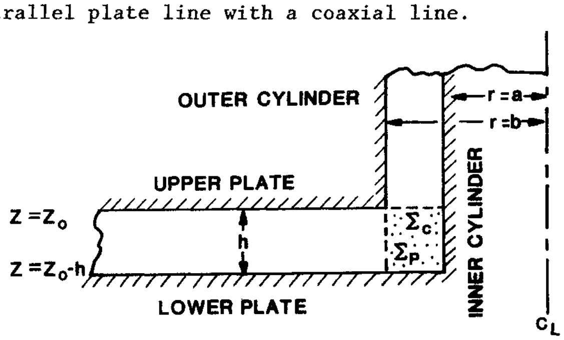

Fig. 1. The junction of a parallel plate line with a coaxial line. The structure is a figure of revolution about the dashed center line, marked C_L.

In the region for which $r > b$ and $z_0 - h < z < z_0$ the fields are given by Eqs. (1.4). When $z > z_0$ and $a < r < b$ the fields are given by Eqs. (1.13). The region $z_0 - h < z < z_0$, $a < r < b$, shown dotted in Fig. 1, is a region common to both lines. It is bounded by the two surfaces Σ_p and Σ_c, as shown in the figure, by the plane $z = z_0 - h$ and by the cylinder $r = a$. In this last region the B-lines bend so that a B_r in the parallel plate section becomes a B_z in the coaxial section.

To work out the details of the transition region we would need to find expressions like Eqs. (1.4) and (1.13) for this region and match the fields smoothly. We shall not do this but shall rather make use of the fact that this troublesome region is small ($h/a \ll 1$).

First of all, we can equate the two expressions for B_θ evaluated at the corner $z = z_0$, $r = b$ obtaining

$$A_m(b/r_0)^m - B_m(r_0/b)^m = C_m e^{mz_0/\rho} - D_m e^{-mz_0/\rho} \quad . \tag{2.1}$$

We need one more boundary condition which we shall express as flux conservation. First of all we consider the surfaces Σ_p and Σ_c to be extended a distance δ, the skin depth, into the metal walls. The surface areas included within the small angle $d\theta$, after this extension, will be

$$\begin{aligned} d\Sigma_p &= b(h + 2\delta)\ d\theta \quad , \\ d\Sigma_c &= \tfrac{1}{2}[(b + \delta)^2 - (a - \delta)^2]\ d\theta \quad , \end{aligned} \tag{2.2}$$

where ρ is the mean radius given by Eq. (1.10). We have assumed that the skin thickness is small enough that it is not influenced by the curvature of the cylindrical surfaces. Our flux conservation condition becomes

$$-B_r^{(p\ell)}\ d\Sigma_p = B_z^{(cyl)}\ d\Sigma_c \quad . \tag{2.3}$$

The minus sign arrises because an inward (i.e., negative) radial flux continues as an upward (i.e., positive) z-flux. From Eqs. (1.4), (1.13) and (2.2) we see that Eq. (2.3) becomes

$$A_m(b/r_0)^m + B_m(r_0/b)^m = -C_m e^{mz_0/\rho} - D_m e^{-mz_0/\rho} \quad . \tag{2.4}$$

Note that the finite skin thickness cancelled out exactly.

The boundary conditions (2.1) and (2.4) give

$$\begin{aligned} A_m &= -D_m(r_0/b)^m\ e^{-mz_0/\rho} \quad , \\ B_m &= -C_m(b/r_0)^m\ e^{mz_0/\rho} \quad . \end{aligned} \tag{2.5}$$

With these equations we can join together parallel plate lines and coaxial lines. We shall illustrate this in the next section.

Compound Structure

In this section we shall consider a stepped coaxial line of n sections driven by a parallel plate line which in turn is asymmetrically driven by a current source. In Fig. 2 we depict such a structure with three steps, i.e., n = 3. We shall, however, keep n arbitrary, as this causes no complication. The coaxial lines and parallel plate sections will be numbered 1, 2, ..., n from top to bottom; parallel plate line number n is the line which drives the whole structure. The total length of the stepped coaxial line we denote by L; thus $L = \ell_1 + \ell_2 + \cdots \ell_n$. The quantities a_s, b_s, ℓ_s ($s = 1, 2, \cdots, n$) refer respectively to the inner radius, the outer radius, and the length of the $s^{\underline{th}}$ coaxial section, and $\rho_s \equiv \tfrac{1}{2}(a_s + b_s)$. We shall denote coefficients in the $s^{\underline{th}}$ coaxial section by $C_m^{(s)}$,

$D_m^{(s)}$ and in the $s^{\underline{th}}$ parallel plate section by $A_m^{(s)}$, $B_m^{(s)}$.

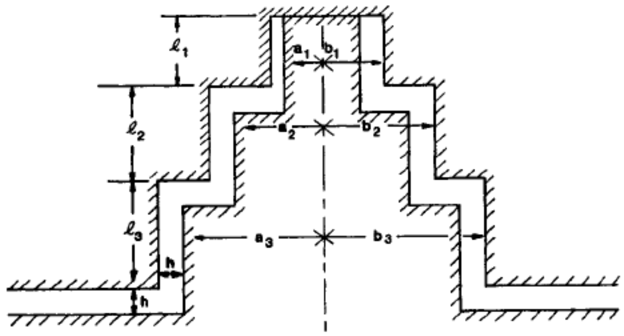

Fig. 2. A stepped coaxial line joined to a parallel plate line. The structure is a figure of revolution about the dot-dashed center line.

We specify that there is a short circuit at the top of coaxial line 1, i.e., at the very top of the structure where $z = L$. At this point B_z must vanish, thus Eq. (1.13) gives

$$C_m^{(1)} = E_m e^{-mL/\rho_1} \quad , \quad D_m^{(1)} = - E_m e^{mL/\rho_1} \quad , \qquad (3.1)$$

where E_m is a new set of constants (time functions) introduced for convenience. Let us now join this shorted coaxial line with the first parallel plate section, this to the second coaxial section and so on. By induction we obtain the result

$$A_m^{(n)} = E_m (r_0/b_1)^m \, e^{+m\gamma_n} \quad ,$$

$$B_m^{(n)} = E_m (b_1/r_0)^m \, e^{-m\gamma_n} \quad , \qquad (3.2)$$

$$\gamma_n \equiv \sum_{s=1}^{n} \ell_s/\rho_s$$

For our compound structure, the fields in the driving line, parallel plate section n, are

$$B_r = \sum_{m=1}^{\infty} (m/r) E_m (r/b_1)^m \, e^{m\gamma_n}$$

$$[1 - (b_1/r)^{2m} \, e^{-2m\gamma_n}] \sin m\theta \quad , \qquad (3.3)$$

$$B_\theta = 2I/r + \sum_{m=1}^{\infty} (m/r) E_m (r/b_1)^m \, e^{m\gamma_n}$$

$$[1 + (b_1/r)^{2m} \, e^{-2m\gamma_n}] \cos m\theta \quad , \quad B_z = 0 \quad .$$

At the very top of the stepped coaxial structure, at the short circuit on coaxial line 1, the B-field evaluated on the surface of the inner conductor is

$$B_r = 0 \quad , \quad B_z = 0 \quad ,$$
$$B_\theta = 2I/a_1 + \sum_{m=1}^{\infty} 2(m/a_1) E_m \cos m\theta \quad . \qquad (3.4)$$

Our final task will be evaluate the quantities E_m.

The true situation we wish to model is fed by a pair of rectangular parallel plates. With our formalism it is very difficult to handle such shapes. Therefore, we shall make an assumption which appears reasonable. Namely, we shall assume that we preserve the essential features of the problem if our feed line consists of a pair of parallel circular discs, concentric with the stepped coaxial line, and if we then feed these discs in a realistically asymmetric fashion. Accordingly our feed line consists of two discs of radius b_0.

The boundary condition at the conducting surfaces of the parallel plate line is that

$$B_\theta = \begin{cases} 4\pi \, K_r \text{ on the upper plate} \\ -4\pi \, K_r \text{ on the lower plate} \end{cases} \quad , \qquad (3.5)$$

where K_r is the radial component of the surface current density. We shall specify the ultimate drive of our system as a prescribed radial surface current density injected over the perifery of the parallel plate line, $r = b_0$. Thus

$$K_r = F(t,\theta) \text{ on } r = b_0 \quad , \qquad (3.6)$$

where F is a known function of t and θ, so we can Fourier expand writing

$$F(t,\theta) = \sum_{m=0}^{\infty} F_m(t) \cos m\theta \quad . \qquad (3.7)$$

According to Eq. (3.5) we compare Fourier coefficients of Eqs. (3.3) and (3.7) and obtain

$$2I/b_0 = 4\pi F_0 \qquad (3.8)$$

$$E_m = (4\pi b_0/m) F_m \left[\frac{(b_1/b_0)^m \, e^{-m\gamma_n}}{1 + (b_1/b_0)^{2m} \, e^{-2m\gamma_n}} \right] .$$

If we substitute this into Eq. (3.4), we see that the total B-field at the top of the stepped coaxial line is given by

$$B = B_\theta = 2I/a_1 + \frac{8\pi}{a_1} \sum_{m=1}^{\infty} \left[\frac{F_m (b_1/b_0)^m \, e^{-m\gamma_n}}{1 + (b_1/b_0)^{2m} \, e^{-2m\gamma_n}} \right] \cos m\theta \quad . \qquad (3.9)$$

The infinite sum, which represents the asymmetry in the B-field at the top of the compound coaxial system is seen to be small, for normally b_1/b_0 will be quite small and γ_n will be noticeably greater than one. Because of the exponential dependence of the coefficients of cos mθ on m, only the first few terms will be important.

We conclude this section with a few concrete examples of asymmetric input current distributions.

First consider the case

$$F(t,\theta) = \begin{cases} K(t) \text{ for } \pi - \theta_0 < \theta < \pi + \theta_0 \quad \text{, and} \\ 0 \text{ otherwise} \quad . \end{cases} \tag{3.10}$$

We are driving the parallel plate line by a current which is uniform over a circular arc of width $w = 2b_0\theta_0$, with a total current $I = 2b_0 \theta_0 K(t)$. Fourier expanding Eq. (3.10) gives

$$F_0 = \frac{\theta_0 K}{\pi} \quad ,$$
$$F_m = (-)^m \frac{2K}{m\pi} \sin m\theta_0 \quad , m \neq 0 \quad . \tag{3.11}$$

With these values for the F_m, Eq. (3.9) becomes

$$B_\theta = \frac{2I}{a_1} \left\{ 1 + \sum_{m=1}^{\infty} (-)^m \left(\frac{4 \sin m\theta_0}{m\theta_0} \right) \left[\frac{(b_1/b_0)^m e^{-m\gamma_n}}{1+(b_1/b_0)^{2m} e^{-2m\gamma_n}} \right] \cos m\theta \right\}, \tag{3.12}$$

at the top of the compound coaxial line.

As a second example we consider a symmetric two sided feed. Thus we set,

$$F(t,\theta) = \begin{cases} \tfrac{1}{2}K(t) \text{ for } \pi - \theta_0 < \theta < \pi + \theta_0 \text{ and for} \\ \theta_0 < \theta < -\theta_0 \quad \text{, and} \\ 0 \text{ otherwise} \quad . \end{cases} \tag{3.13}$$

Now our Fourier coefficients are

$$F_0 = \frac{\theta_0 K}{\pi} \quad ,$$
$$F_m = \frac{K}{m\pi} [1 + (-)^m] \sin m\theta_0 \quad , m \neq 0 \quad . \tag{3.14}$$

For even values of m this is identical with Eqs. (3.11) whereas for odd values of m, F_m vanishes. Thus, for the field at the top of the coaxial line we recover Eq. (3.12) modified solely by deleting from the summation all terms for which m is odd.

Generally, for q equally spaced feed strips we simply delete from Eq. (3.12) those terms for which m is not divisible by q.

Discussion of the Asymmetry

As an illustration of the foregoing analysis, let us consider a numerical example, which is an early design for one of our foil implosion experiments. The foil is the very top section (line 1) of Fig. 1. The field is given by Eq. (3.12); B_r and B_z vanish here. Our stepped coaxial line consists of three sections, whose dimensions are given in Table 1.

Table 1.
Dimensions, in cm., of the Stepped Coaxial Line

i	a_i	b_i	ρ_i	ℓ_i	ℓ_i/ρ_i
1	4.2	4.7	4.45	5.5	1.2340
2	8.2	8.7	8.45	3.4	0.4024
3	11.0	11.5	11.25	4.8	0.4267

For this line n = 3 and h = 0.5 cm. From the table we get $\gamma_n \equiv \gamma_3 = 2.063$ and $e^{-\gamma_n} = 0.1271$.

The two feed plates are square, 76.2 cm (30 in.) on a side and are symmetrically fed over the full width of two opposite sides. Thus θ_0, the half angle of one of the feed strips, is $\pi/4$, whence

$$\frac{\sin \theta_0}{\theta_0} = 0.9003 \quad \text{and} \quad \frac{\sin 2\theta_0}{2\theta_0} = 0.6366 \quad . \tag{4.1}$$

We must, for our model, replace the square feed plates by circular discs. We conservatively take b_0 = 38.1 cm., i.e., we replace the squares by inscribed circles. This gives $b_1/b_0 = 0.1234$ whence

$$X \equiv (b_1/b_0)e^{-\gamma_n} = 1.57 \times 10^{-2} \quad ,$$
$$X^2 = 2.46 \times 10^{-4} \quad . \tag{4.2}$$

For such a geometry we see that all the denomenators in Eq. (3.12) can be replaced by unity and the first three terms become

$$B_\theta = \frac{2I}{a_1} \{1 - (\frac{4 \sin \theta_0}{\theta_0})X \cos \theta + (\frac{4 \sin 2\theta_0}{2\theta_0})X^2 \cos 2\theta \ \ldots \} \tag{4.3}$$

if we drive the parallel plates from one side only.

The importance of the field asymmetry is that the consequent asymmetry of the magnetic pressure will drive a Rayleigh-Taylor instability. The driving pressure is $P = B_\theta^2/8\pi$. Using the above numerical values we would have

$$P = \frac{I^2}{2\pi a_1{}^2} \{1 - 0.1131 \cos \theta + (2.852 \times 10^{-3}) \cos 2\theta \cdots\} \tag{4.4}$$

if the line were driven from one side only.

For the balanced, two sided drive, we must omit the terms of Eq. (4.3) in cos θ, cos 3θ etc. In this case we obtain

$$P = \frac{I^2}{2\pi a_1{}^2} \{1 + (1.253 \times 10^{-3}) \cos 2\theta + \cdots\} \tag{4.5}$$

From the value of X in our case, Eq. (4.2), it is clear that the further terms in Eqs. (4.4) and (4.5), proportional to $X^3 \cos 3\theta$, $X^4 \cos 4\theta$ etc., are completely negligible.

Having calculated the pressure asymmetry, Eq. (4.4) or (4.5), we have the input data for our principal problem, namely describing the resulting Rayleigh-Taylor instability. This, however, is the subject for a different paper.